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Interference patterns in spiral wave drift induced by a two-point feedback

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The drift velocity field describing spiral wave motion in an excitable medium subjected to a two-point feedback control is derived and analyzed. Although for a small distance d between the two measuring points a discrete set of circular shaped attractors are observed, an increase of d induces a sequence of global bifurcations that destroy this attractor structure. These bifurcations result in the appearance of smooth unrestricted lines with zero drift velocity, similarly to zero intensity lines under destructive interference in linear optics. The existence of such unusual equilibrium manifolds is demonstrated analytically and confirmed by computations with the Oregonator model as well as by experiments with the light-sensitive Belousov-Zhabotinsky reaction.

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Spiral waves represent a typical example of self-organized processes in excitable media of quite different nature starting from physics to chemistry and biology [1]. In addition to observations and simulations of spiral waves in homogeneous and stationary media, external forcing is very efficiently used as an "active" tool of study [2]. Moreover, feedback-mediated forcing provides the opportunity to control spiral wave location, which is especially important for such an application as the low voltage defibrillation of cardiac tissue [3].

In particular, if a feedback signal is determined as an integral of wave activity over a certain spatial domain within an excitable medium, a spiral wave drift can be induced [4–7]. In that case the resulting drift velocity can be found as a sum of single drift vectors induced separately by each point of the integration domain [8]. Note that a similar superposition principle is valid for linear waves and results in interference patterns, often containing lines of destructive interference. Hence we can expect the existence of unusual one-dimensional equilibrium manifolds in a feedback-induced drift velocity field.

In this paper we demonstrate the existence, clarify the underlying physics and indicate basic consequences of an interference in the drift velocity fields mediated by feedback in nonlinear excitable media. To this aim we derive the drift velocity field generated by a feedback taken from two measuring points, perform numerical integrations of the Oregonator model, and confirm the obtained theoretical results by the experiments carried out in the light-sensitive Belousov-Zhabotinsky (BZ) reaction.

In order to introduce the feedback algorithm let us consider the modified Oregonator model of a thin layer of the light-sensitive BZ solution

$$\frac{\partial u}{\partial t} = \nabla^2 u + \frac{1}{\epsilon}\left[u - u^2 - (fv + I)\frac{u-q}{u+q}\right],$$

$$\frac{\partial v}{\partial t} = u - v. \qquad (1)$$

Here the variables u and v correspond to the concentrations of the autocatalytic species $HBrO_2$ and the oxidized form of the catalyst, respectively. The parameters $\epsilon=0.05$, $q=0.002$, and $f=2.0$ were fixed. The term $I=I(t)$ describes the bromide production proportional to the external illumination [9]. To insert a feedback control the illumination intensity $I(t)$ is computed as

$$I(t) = I_0 + k_{fb}[B(t-\tau) - B_0], \qquad (2)$$

where τ is the time delay, k_{fb} is the feedback gain, and $I_0 = 0.01$ and $B_0=0.06$ are constants. $B(t)$ specifies the concentration $v(x,y,t)$ averaged over the integration domain S of area s,

$$B(t) = \frac{1}{s}\int_S v(x',y',t)dx'dy'. \qquad (3)$$

Let us assume first that $k_{fb}=0$, i.e., the feedback loop is not closed. Then $I(t)=I_0$ and the value $B(t)$ is determined by the location of the spiral core center (x,y): $B(t)=B(t|x,y)$. Below rigidly rotating or slightly meandering spiral waves are considered. In this case the value $v(x',y',t|x,y)$ measured at a site (x',y') located sufficiently far away from the core center oscillates at the period T_∞. Hence, the corresponding feedback signal $I(t|x,y)$ determined from Eqs. (2) and (3) is also periodic in time and its first Fourier component reads

$$I_1(t|x,y) = k_{fb}A(x,y)\cos[\omega t - \omega\tau - \phi(x,y)], \qquad (4)$$

where $\omega=2\pi/T_\infty$, and the amplitude $A(x,y)$ and the phase $\phi(x,y)$ are defined by the following expression:

$$A(x,y)e^{i\phi(x,y)} = \frac{2}{T_\infty}\int_0^{T_\infty} B(t|x,y)\exp(i\omega t)dt. \qquad (5)$$

The periodic forcing $I(t)=I_1(t|x,y)$ with $k_{fb}>0$ applied to system (1) induces a spiral wave drift [4–8]. If this drift is slow, then the actual location of the core center (x,y) determines its velocity $V(x,y)\sim k_{fb}A(x,y)$ and direction with respect to the x axis

$$\gamma(x,y) = \varphi + \omega\tau + \phi(x,y). \qquad (6)$$

The constant φ specifies the direction of the drift induced in the case $\tau=0$ and $\phi=0$. For the model (1)–(3) with the parameters indicated above $\varphi = -0.5$ [8]. Hence the drift velocity field can be written as

$$\frac{dx}{dt} = V(x,y)\cos\gamma(x,y), \quad \frac{dy}{dt} = V(x,y)\sin\gamma(x,y). \tag{7}$$

If the shape of a slightly meandering wave can be approximated by a counterclockwise rotating Archimedean spiral, the first Fourier component of $v(x',y',t|x,y)$ reads

$$v_1(x',y',t|x,y) = v_m \cos(\theta + 2\pi r/\lambda - \omega t), \tag{8}$$

where v_m is a constant, λ is the spiral wavelength, and r and θ are polar coordinates,

$$r = \sqrt{(x'-x)^2 + (y'-y)^2}, \quad \theta = \arctan\left(\frac{y'-y}{x'-x}\right). \tag{9}$$

Substituting Eqs. (3) and (8) into Eq. (5), we get

$$A(x,y)e^{i\phi(x,y)} = \frac{v_m}{s}\int_S \exp[i\Phi(x',y'|x,y)]dx'dy', \tag{10}$$

where

$$\Phi(x',y'|x,y) = \theta + \frac{2\pi}{\lambda}r. \tag{11}$$

It is important to stress that if the size of the domain S is much smaller than the spiral wavelength λ, we get the limiting case of one-point feedback control, which is specified by Eqs. (9)–(11) with $(x',y')=(0,0)$ [8]. In this case $\phi(x,y)=\Phi(0,0,|x,y)$ and Eq. (6) determines the drift direction $\gamma(x,y)$ for given values τ and φ. In the general case, the resulting drift velocity field is derived using Eqs. (9)–(11) and (6) as a sum of all drift vectors induced by single points composing an integration domain of arbitrary shape.

Below the case of the two-point feedback is considered, where the feedback signal is taken from two measuring points $(x'_+,y'_+)=(0,a)$ and $(x'_-,y'_-)=(0,-a)$. Substituting these coordinates into Eq. (11), we get two functions $\Phi_+(x,y)$ and $\Phi_-(x,y)$ describing the influence of the feedback taken from each point separately

$$\Phi_\pm(x,y) = \arctan\left(\frac{\pm a - y}{-x}\right) + \frac{2\pi}{\lambda}r_\pm, \tag{12}$$

where $r_\pm = \sqrt{x^2 + (a \mp y)^2}$. The amplitude $A(x,y)$ and the phase $\phi(x,y)$ of the drift velocity field induced by the two points together are determined from Eq. (10) as

$$A(x,y)e^{i\phi(x,y)} = [\exp(i\Phi_+) + \exp(i\Phi_-)]v_m/2, \tag{13}$$

and then the drift angle $\gamma(x,y)$ follows from Eq. (6).

It can be easily seen from Eq. (13) that the amplitude $A(x,y)$ vanishes if

$$\Phi_+(x,y) - \Phi_-(x,y) = \pi(2m+1), \tag{14}$$

where m is an integer. A solution of this equation specifies a smooth line or a set of lines on the (x,y) plane.

To analyze the obtained drift velocity field it is suitable to choose the distance between two points $d=2a$ as a control parameter. If the distance $d/\lambda \ll 1$, the drift velocity field is very similar to that induced by the one-point feedback studied in [8]. It includes a set of circular-shaped attracting manifolds called resonance attractors [10]. This attractor structure still exists for any $d/\lambda < 0.5$. For example, the drift velocity field obtained for $d/\lambda = 0.45$ is shown in Fig. 1. The thick solid line represents numerical results obtained for the model (1)–(3) and illustrates the existence of a circular-shaped resonance attractor in quantitative agreement with the drift velocity field predicted analytically. However, in contrast to one-point feedback, the magnitude of the drift vectors is not a constant, but is very slow in the upper and lower parts of the attractors. Moreover, the drift velocity vanishes at a smooth curve connecting the measuring points. It is natural to refer to such an unusual equilibrium manifold as a fixed line, in analogy to well-studied fixed points.

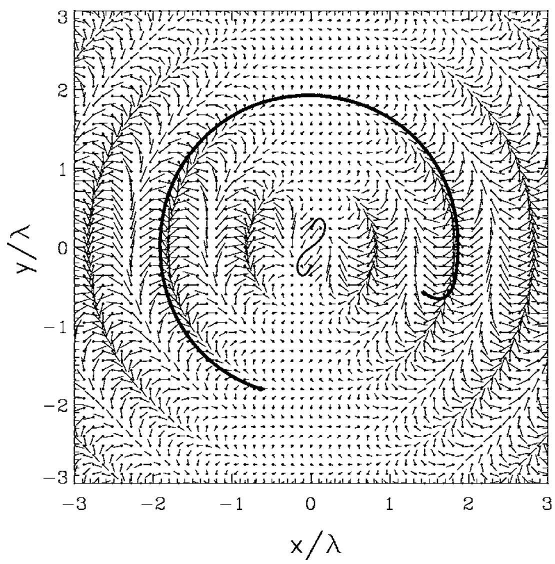

FIG. 1. Drift velocity field determined by Eqs. (6) and (13) for $d/\lambda = 0.45$. The fixed line (thin solid) satisfies Eq. (14). The thick solid line represents the core center trajectory computed for the model (1)–(3) with $k_{fb}=0.02$ and $\tau=0$.

The drift velocity field changes dramatically for $0.5 \leq d/\lambda < 1.5$. In this case there are three equilibrium manifolds, which are unrestricted in space. Thus, the circular-shaped attractors existing for $d/\lambda < 0.5$ are destroyed as shown in Fig. 2. In accordance with the predicted velocity field, the drift of a spiral wave core, computed for the Oregonator model, first follows an approximately circular trajectory and then stops somewhere inside the medium. In the case shown in Fig. 2(a) the drift practically stops relatively far away from the fixed line, since the drift velocity is very slow in a broad region surrounding this line. In Fig. 2(b) variations in the velocity magnitude are rather sharp in the vicinity of the fixed lines, therefore the drift stops practically at the predicted lines.

We complemented our study by experiments performed with the light-sensitive version of the BZ reaction using an open-gel reactor [11]. The reaction takes place in a thin (0.5 mm thickness) silicahydrogel layer of diameter 63 mm, where the catalyst is immobilized. Premixed feeding solution prepared as indicated in [7] is pumped continuously through the reactor. To protect the active layer from stirring effects, it

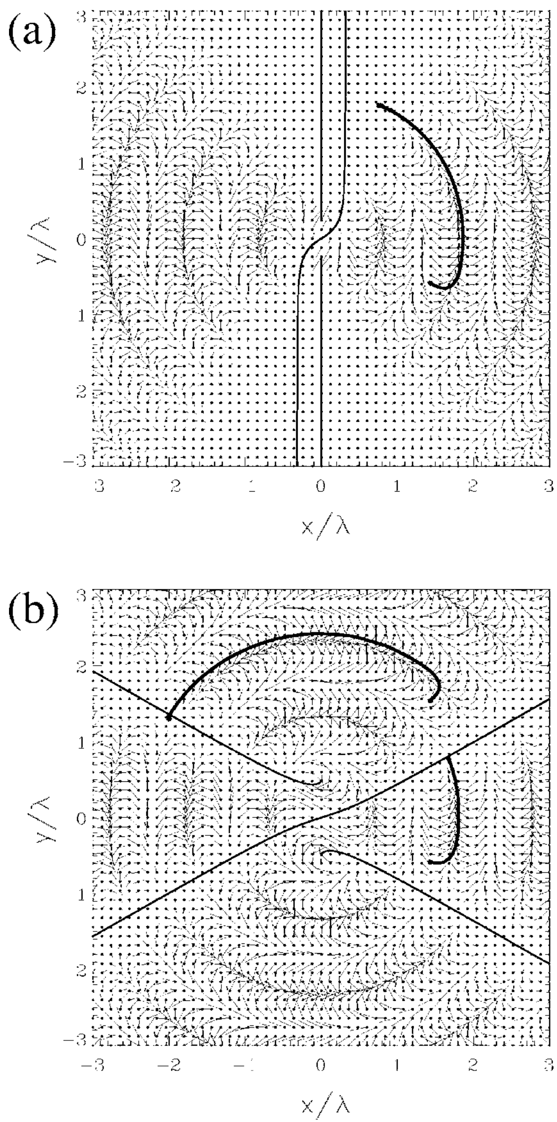

FIG. 2. Drift velocity fields obtained from Eqs. (6) and (13) for (a) $d/\lambda=0.5$, (b) $d/\lambda=1.0$. The fixed lines (thin solid) satisfy Eq. (14). Thick solids represent the trajectories of the core center computed for the model (1)–(3) with $k_{fb}=0.02$ and $\tau=0$.

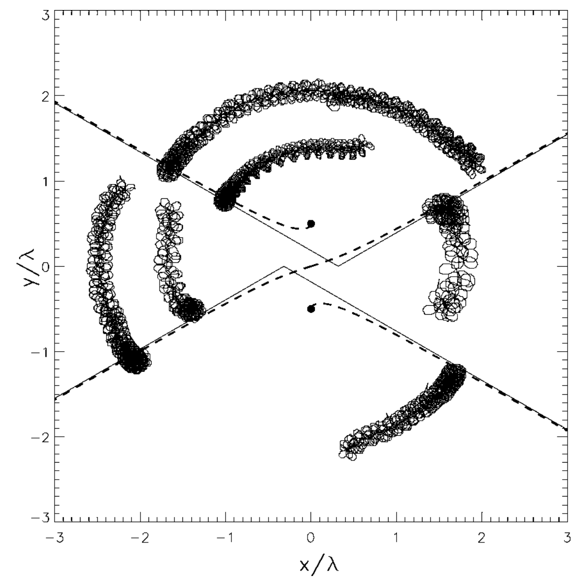

FIG. 3. Trajectories of spiral wave tips observed in the experiments with the light-sensitive BZ reaction. Two measuring points are shown by black dots. Dashed lines depict fixed lines determined numerically from Eq. (14). Straight asymptotes correspond to analytical expressions (17)–(19).

is covered by an inactive gel layer not loaded with the catalyst. The active layer is illuminated by a video projector (Panasonic PT-L555E) controlled by a computer. Every one second the images of the oxidation waves appearing in the gel layer are detected in transmitted light by a charge-coupled device (CCD) camera (Sony AVC D7CE) and digitized for immediate processing by the computer. During the same time step the signal controlling the projector can be changed in accordance with the processed information to close the feedback loop.

A single spiral wave is created in the gel disk by breaking a wave front with a cold intense light spot. The location of the spiral wave tip is defined online as the intersection point of contour lines ($0.6\times$amplitude) extracted from two digitized images taken with time interval 2.0 s. An unperturbed spiral has a wavelength $\lambda\approx 2.0$ mm. Its tip describes a meandering trajectory containing about four lobes. The rotation period measured far away from the symmetry center is $T_\infty\approx 40$ s.

In our experiments the feedback signal is determined in accordance with Eq. (2), where the value $B(t)$ is the sum of the intensities of the transmitted light measured at two points located at the distance $d=\lambda$. The results of six experiments with different initial locations of the spiral core are shown in Fig. 3. In full agreement with the predicted drift velocity field shown in Fig. 2(b), the drift of spiral waves stops at the fixed lines.

Thus, the velocity field for spiral core drift induced by the two-point feedback is determined analytically by Eqs. (6), (7), (12), and (13). It contains a set of fixed lines instead of fixed points commonly discussed in the theory of structurally stable dynamical systems [12].

The analysis of Eq. (14) reveals that one fixed line passes through the origin of the coordinate system and can be approximated here by $y=\lambda/(\pi d)x$, cf. Figs. 1 and 2. If $d/\lambda<0.5$ this manifold connects two measuring points and is restricted in space. New unrestricted equilibrium manifolds appear at $d=d_n$, where

$$d_n=\lambda(0.5+n),\quad n\geqslant 0. \tag{15}$$

Close before these bifurcations occur, the fixed line in a vicinity of the point $(0,a)$ looks like a strongly curved loop (see Fig. 1) with a top located approximately at

$$x_m=\lambda/(2\pi),\quad y_m=x_m/\sqrt{2(1-d/d_n)}. \tag{16}$$

The ordinate y_m grows with d and diverges exactly at these bifurcation points. The loop breaks down and transforms into two unrestricted lines. One line is $x=0$, and the second one approaches $x=\lambda/\pi$ for $y\to\infty$, as illustrated in Fig. 2(a).

Similar transformation occurs with the loop of the fixed line starting at $(0,-a)$.

For $d/\lambda > 0.5$ the fixed lines approach the straight asymptotes

$$y = (x - x_0)\tan\beta, \tag{17}$$

$$d\sin\beta = \lambda(0.5 + n), \tag{18}$$

$$x_0 = \lambda/(2\pi\sin\beta). \tag{19}$$

These asymptotes are in perfect agreement with the numerical solution of Eq. (14), cf. Fig. 3.

There is a very close analogy between the unusual one-dimensional equilibrium manifold reported here and the lines of the destructive interference appearing due to linear superposition of concentric waves emitted by two point sources. Orientations of these lines also obey Eq. (18). A shift x_0 specified by Eq. (19) is due to an arctangent term in (12), specific for spiral-shaped waves.

Not only the location of the fixed lines, but also the drift directions predicted by Eqs. (6) and (13) closely correspond to numerical and experimental data. This is because the Archimedean spiral approximation (8) is very precise, if $r > r_A$. For given parameters in the Oregonator model $r_A \approx 0.2\lambda$ (cf. Ref. [8]), i.e., it is small with respect to the radius of the resonance attractor. If the core center is located too close to a measuring point, strong deviations from the computed drift velocity field can be expected up to the appearance of the entrainment or asynchronous attractors [8].

It follows from Eq. (13) that the phase $\phi(x,y)$ increases when core center (x,y) is shifted along a fixed line away from the origin. Hence, due to Eq. (6), the drift direction rotates during such a shift. It can be seen also that the drift direction jumps to the opposite one by crossing the fixed line. Therefore, attracting and repelling segments alternate along the fixed line with the spatial period λ.

In summary, the two-point feedback is proven to be very suitable to study basic features of spiral wave drift induced by a feedback control in excitable media. The results of the analytical study are in agreement with the numerical and experimental data and reveal the existence of unusual global bifurcations leading to the appearance of unrestricted fixed lines with zero drift velocity. It is important to take into account such phenomena in future studies of feedback controlled systems with other geometries of the integration domains (e.g., elliptical or rectangular ones).

Generally speaking, fixed lines are expected in structurally unstable dynamical systems usually considered as unrealistic [12]. However, our study gives an example of a real system, where unavoidable experimental noise does not qualitatively alter the phase portrait of the model system (6), (7), (12), and (13). This motivates future study of the possible behavior of a second order dynamical system near a fixed line, which is a challenge for dynamic system theory [13]. It is of interest also to find out other dynamical systems with similar unusual properties, where interference patterns determine the phase portrait.

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